Discover and Design Sun Twisty Puzzles

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Abstract

This article presents the algorithm of rigorous searching for non–trivial twisty puzzles with sun mechanism. It also presents some hints of how to design their mechanism.

1 Introduction

Currently, there are several twisty puzzles with sun mechanism, which we will call *sun twisty puzzles* or simply *sun puzzles*. Several other puzzles are custom designed and made. Figure 1 presents commercially available puzzles of cubic form, Figure 2 presents commercially available dodecahedral family of puzzles from mf8.

Figure 1: Sun puzzles with cubic shape: left – DaYan Bagua Cube, right – mf8 Sun Cube.

Sun mechanism allows face twists by half of usual angle: 45° instead of 90° for cubic form and 36° instead of 72° for dodecahedral forms[1].

The usual sun mechanism is presented in Figure 3 applied to Sun Cube[2]. We will name the pieces of sun mechanism as follows:

Center — not shown,

Vertex — left and right, gray background,

[1]WitEden Mixup series as well as mf8 Son–Mum series of puzzles allow twists by half of usual angle of the middle layer. These new moves form different combinations and are realized by different mechanism.

[2]All 3D diagrams are interactive. They can be interactively viewed with Adobe Reader ®.

Figure 2: mf8 family of sun puzzles: 1st row left to righ – Sunminx and Cullinan, 2nd row left to right – Sky Eyes and Big Dipper.

Edge — middle, gray background,

Left Ray — light blue background, bottom left and top right to edge,

Right Ray — light green background, top left and bottom right to edge,

Petal — pink background, 3 top and 3 bottom are shown,

Hidden — light yellow background, between edge and vertices.

On Bagua mechanism, hidden pieces are actually visible and have colored stickers.

Petal and hidden pieces both have mirror symmetry. Left and right rays (mirror of each other) look identical and symmetrical on the puzzle surface, however in interior of puzzle they are different. The cluster composed of a ray (left or right) and a hidden part make a mirror symmetrical composite.

Because of half turns, the sun mechanism allows interchanging of vertices with edges. More than that, edges can be rotated 3-fold (like vertices) and vertices in pace of edges can be rotated 2-fold (like edges).

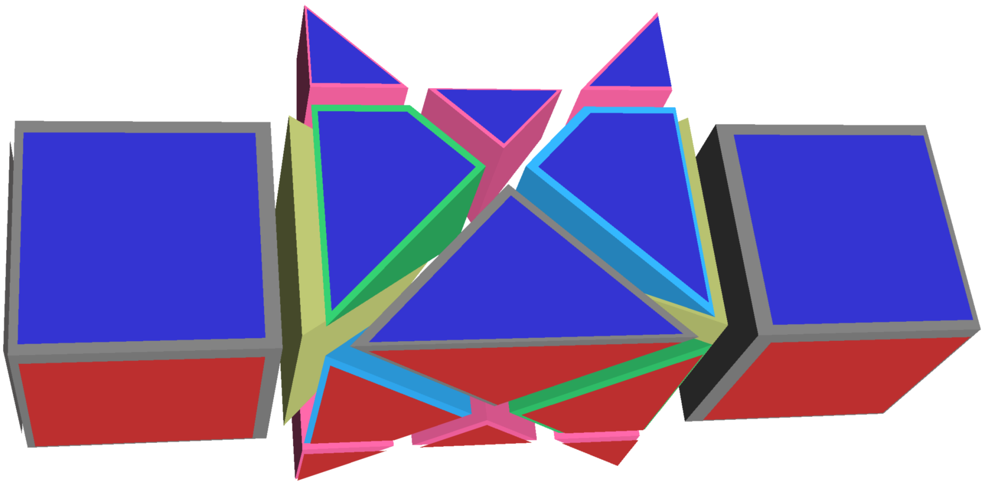

Figure 3: Elements of mechanism of Sun Cube.

2 Definitions

Twisty puzzles have rotational symmetry. The number of turns by minimal angle around an axis, that make up whole turn is named *order of axis*. Sun mechanism doubles the order of axes.

The half–twists, specific to sun puzzles, increase their complexity. We define *combinatorial symmetry* of a puzzle as the maximal number of interchangeable pieces multiplied by their symmetry (number of ways a piece can be placed) across all part types. In case of Sun Cube, the combinatorial symmetry can be computed as:

Centers: $6 \times 8 = 48$,

Petals: $48 \times 1 = 48$,

Left or right rays: $24 \times 1 = 24$,

Edges: $12 \times 2 = 24$,

Hiddens: $24 \times 1 = 24$,

Vertices: $8 \times 3 = 24$.

Among these numbers, the largest is 48. This is twice as much as for Cube 3×3, whose combinatorial symmetry is 24. In case of Cube 3×3, combinatorial and geometrical symmetries match: the cube has a group of even symmetry[3] of order 24.

The combinatorial symmetry of Sunminx, Sky Eyes, Cullinan and Big Dipper are all equal to 120 (computed by petals). This is twice as much as 60, the combinatorial symmetry of Megaminx 3×3. It is also larger than 96, the combinatorial symmetry of Crazy Comet whose sun variant is Sky Eyes. The combinatoral symmetry of sun puzzles can always be computed using petal pieces and equals to the number of axes multiplied by their order: $6 \times 8 = 48$ for cubic shaped puzzles and $12 \times 10 = 120$ for dodecahedral shaped puzzles.

The pattern on faces of sun puzzles always has the form of *regular star polygons*. These polygons are similar to convex regular polygons, but their sides connect not necessarily adjacent vertices. The mathematical notation for star polygon is a fraction whose numerator indicates the number

[3] The whole group of symmetry contains even rotations and odd mirror motions. The mirror motions are not available in physically produces twisty puzzles.

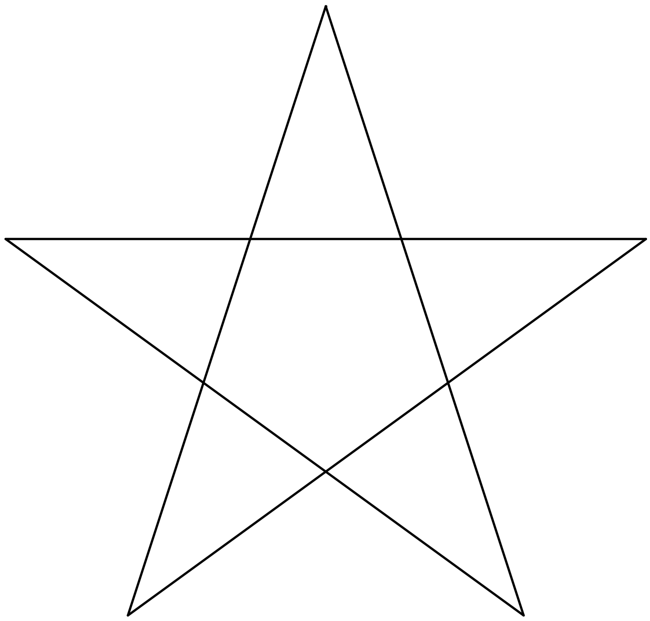

Figure 4: Pentagram as regular star polygon $\frac{5}{2}$.

of vertices and denominator shows how many adjacent vertices are skipped when connecting two vertices. When denominator is 1, no vertices are skipped and the polygon is regular, when it is 2, one adjacent vertex is skipped, when it is 3, two vertices are skipped and so on. Figure 4 shows regular star 5/2-gon[4].

Straight–lined polygons can have at least 3 sides, a 2-gon degenerates to two-sided segment. This is also the case of star polygons: their fractional value cannot be less than or equal to 2. The simplest star polygon is 5/2 and has fractional value 2.5.

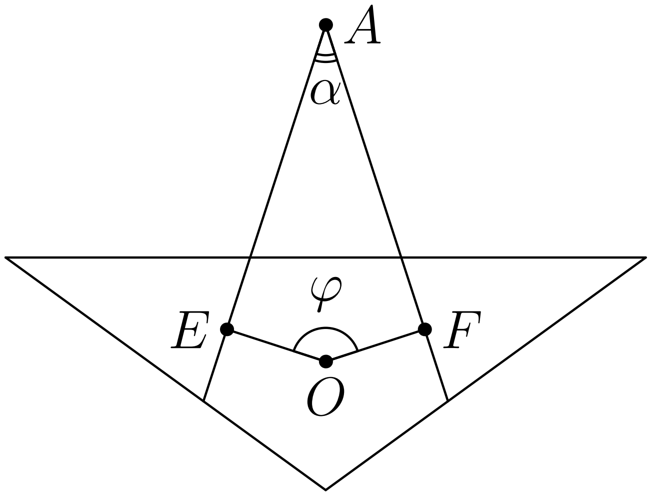

Figure 5: Computing angles of star polygon.

The mathematical notation of star polygons not only indicates their shape but also helps computing their angles (Figure 5). When going from some vertex A of star polygon n/m by its sides to the same vertex, the center is surrounded m times and the path is divided into n equal angles. So,

[4]Sometimes the fraction n/m is reducible. In this case, instead of star polygon, the pattern is a compound of several regular polygons. For example, faces of Sun Cube contain pattern 8/2, which is compound of two squares. For the purpose of this article, we treat this case identically with star polygons.

the central angle:

$$\angle EOF = \varphi = 360 \cdot \frac{m}{n}.$$

Next, we can observe in the quadrilateral $OEAF$ two right angles: $\angle OEA$ and $\angle OFA$. This means the polygon angle

$$\angle EAF = \alpha = 180 - \varphi = 180 - 360 \cdot \frac{m}{n} = 180 \left(1 - 2\frac{m}{n}\right). \tag{1}$$

A polyhedron vertex adds angles of faces less than 360. The difference between 360 and the sum of face angles at this vertex is named *vertex defect*.

It is important to mention the *duality* of polyhedra, which exchange faces with vertices. A polyhedron B dual to some polyhedron A can be constructed by placing its vertices in centers of faces of A. Consider adjacent vertices of B the vertices placed on adjacent faces of A. Construct edges by connecting adjacent vertices of B and then construct faces on each closed polygon that has no vertex inside. The duality property is reciprocal: applying duality twice we obtain initial polyhedron. Tetrahedron is dual to itself, cube and octahedron make a dual pair as well as dodecahedron and icosahedron.

3 Geometrical principles and algorithm

Now we are prepared to discuss two important principles for rigorous searching of sun puzzles.

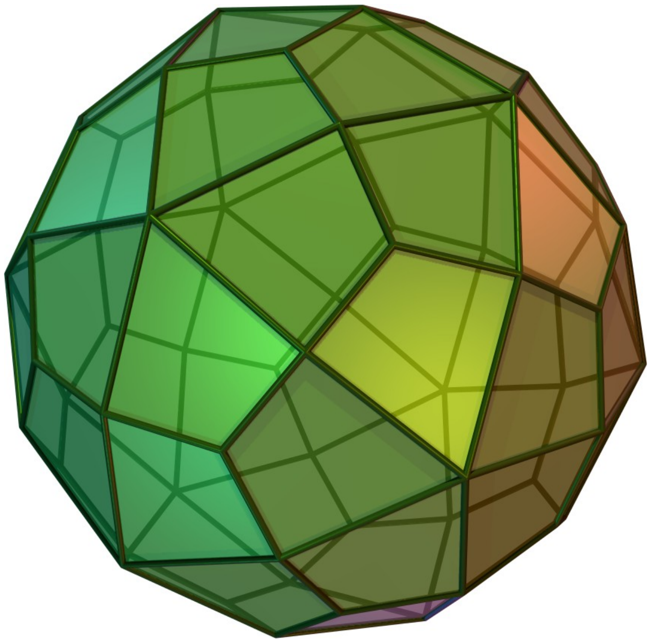

Figure 6: Deltoidal Hexecontahedron.

The first principle: In a sun puzzle with n-fold axes:

- the polygon delimited by center is n-gon,
- the polygon delimited by center and petals is $n/2$-gon,
- the polygon delimited by center, petals and rays is $n/3$-gon,

More than that:

- the angle of puzzle face on 3-fold vertex equal to the angle of $n/2$-gon,
- the angle of puzzle face on 4-fold vertex equals to the angle of $n/3$-gon,
- if there exist 5-fold vertices in sun puzzles, the angle of face on 5-fold vertex is most probably equal to the angle of $n/4$-gon,
- there is probably no 6-fold or higher vertices in sun puzzles.

The claim about the face angle on 5-fold vertex can be sustained by the following fact. When, starting with regular n-gon we *stellate* it, that is, extend its sides in both directions until they are intersected again, we obtain $n/2$-gon whose vertices are further from the center than the vertices of initial polygon. If we extend its sides more, we obtain $n/3$-gon whose vertices are further from the center that the vertices of $n/2$-gon and so on. Similarly, when we have a polyhedron with symmetric vertices, its 3-fold vertices are nearest to the center, 4-fold vertices are further from the center than 3-fold vertices, 5-fold vertices are further than 4-fold vertices and so on. We can observe this behavior in Deltoidal Hexecontahedron in Figure 6 which has regular 3-fold, 4-fold and 5-fold vertices.

Figure 7: Left – LanLan Face Turning Octahedron, right – mf8 Radiolarian Icosaix V3.

When the vertices of puzzle polyhedron are regular, the 4-fold vertices look like in LanLan Face Turning Octahedron and 5-fold ones as in mf8 Radiolarian Icosaix V3 in Figure 7. However, no sun puzzle is known to have 4-fold or 5-fold vertices. Instead of 4-fold vertices, there are clusters of four 3-fold vertices as in Sky Eyes. For the purpose of this article, we can treat clusters of 4 vertices as one 4-fold vertex.

The second principle: All polyhedra have the sum of vertex defect equal to 720°. If this number is expressed in radians, its value is 4π. Somebody can immediately observe this number equals to the area of unite sphere surface, and this in not a coincidence. The area on sphere surface is computed as angular defect. All the points inside the faces of straight–lined polyhedron have neighborhood identical to flat Euclidean plane, the points in interior of edges have neighborhood composed of two flat half–planes, which is again a flat Euclidean plane. And only finite number of vertices have neighborhood different from Euclidean plane. So, all polyhedron sharpness is inside vertices.

When we know the face polygons, we can compute their angles. When we know the vertex configuration of some polyhedron, we can compute the defect of each vertex. Usually, all vertices have regular configuration of one or several types. This means, the vertex defect of d-fold vertex when the rotational axes are n-fold can be computed by formula (apply equation (1) and the 1st principle: $m = d - 1$):

$$360 - d \cdot 180 \left(1 - 2\frac{d-1}{n}\right) = 180 \left(2\frac{(d-1)d}{n} + 2 - d\right). \tag{2}$$

Now we can estimate the number of vertices, edges and faces.

For example, Sun Cube has square faces, whose angles by (1) are 180(1 - 2/4) = 180/2 = 90°. A vertex connects 3 squares which sum to 3 · 90 = 270°, and its defect is 360 - 270 = 90°. So, a cube by the 2nd principle has 720 / 90 = 8 vertices. Sunminx has pentagonal vertices with angles 180(1 - 2/5) = 180 · 3/5 = 108°. A vertex has 3 regular pentagons which sum to 3 · 108 = 324°, and its defect is 360 - 324 = 36°. So, a dodecahedron has 720 / 36 = 20 vertices.

Algorithm of searching for sun puzzles. We start with fixing the desired axes order n. Then, by (2) we compute the defect of vertices. For 3-vertex it is:

$$\Delta_3 = 180 \left(2\frac{(3-1)3}{n} + 2 - 3\right) = 180 \left(\frac{12}{n} - 1\right).$$

For 4-vertex it is:

$$\Delta_4 = 180 \left(2\frac{(4-1)4}{n} + 2 - 4\right) = 180 \left(\frac{24}{n} - 2\right).$$

For 5-vertex it would be:

$$\Delta_5 = 180 \left(2\frac{(5-1)5}{n} + 2 - 5\right) = 180 \left(\frac{40}{n} - 3\right).$$

We can consider the vertex defect as a "credit" of each vertex and have to find the combination of these defects so they sum up to 720.

First, assume n is even and Δ_3 divides 720. In this case all vertices can be 3-fold and the polyhedron is regular. Then the number of vertices:

$$v_3 = \frac{720}{180(12/n - 1)} = \frac{4}{12/n - 1} = \frac{4n}{12 - n}.$$

Then, each face by the 1st principle is a regular $n/2$-gon. If we count the number of vertices of each face they are $fn/2$, but because at each vertex meet 3 faces, we considered each vertex for 3 times. So the number of faces f and vertices v are related as:

$$v_3 = \frac{f_3 n}{2 \cdot 3} = \frac{4n}{12 - n}.$$

Now, we can compute the number of faces as:

$$\frac{f_3 n}{6} = \frac{4n}{12 - n},$$
$$f_3 = \frac{24}{12 - n}.$$

In a similar way, if the polyhedron is regular having all 4-vertices (octahedron), the number of vertices:

$$v_4 = \frac{720}{180(24/n - 2)} = \frac{4}{24/n - 2} = \frac{2n}{12 - n}.$$

The number of $n/3$-gonal faces, which meet 4 at each vertex (principle 1) can be computed as:

$$v_4 = \frac{f_4 n}{3 \cdot 4} = \frac{2n}{12 - n},$$
$$f_4 = f_3 = \frac{24}{12 - n}. \tag{3}$$

We can observe in (3) that the number of faces f depends on axes order n in the same way for 3-fold and 4-fold vertices. No sun twisty puzzles is known to have polyhedron whose vertices are 5-fold or higher. If such puzzles exist, their face number f depends on axes order n by another relation. But if we limit the puzzles search to polyhedra with 3-fold or 4-fold vertices only, we can compute the number of faces, even if the same polyhedron have different vertex types. Table 1 presents the integer solutions of equation (3):

Table 1: Relation between the axes order and the number of axes / faces of polyhedron.

Axes order n	**No of axes f**	**Examples / remarks**
0	2	There is no axis of order 0, missing symmetry is 1-fold axis
4	3	Some pillow puzzle with dihedral symmetry
6	4	E.g. tetrahedron
8	6	Sun Cube
9	8	E.g. octahedron
10	12	Sunminx, Sky Eyes, Cullinan and Big Dipper
11	24	E.g. pentagonal icosi–tetrahedron or deltiodal icosi–tetrahedron
12	∞	2D puzzle

Second, 4-fold vertices in sun puzzles are usually replaced with clusters of 3-fold vertices containing 4 such vertices. Currently, there is no exception to this rule. If we continue the analogy with pieces credit, then the "exchange rate" between 4-fold and 3-fold vertices is 4:1. There is also an "exchange rate" of 2:1 between 3-vertices and edges, observable in existing sun puzzles. The table 2 on page 20 shows the same relation for other sun twisty puzzles. So, we can introduce the "credit" for vertex and edge pieces. As soon as we find one suitable polyhedron, we can compute its "price" from the number of vertices v and edges e as $v + 2e$ and then we can add vertex and subtract edge pieces[5] or vice–versa keeping the total "price" intact. This way, we can search for new polyhedra.

The number of petal pieces p and the sum of left and right rays r_l, r_r is always the same and equal to combinatorial symmetry, the number of faces f multiplies by the axes order n:

$$p = r_l + r_r = fn = \frac{24n}{12 - n}.$$

The number of hidden pieces h equals to the number of left or right rays if the rays are all paired (paired rays belong to the same edge). Otherwise, it is less than their sum by twice the number of edges (2 hidden pieces belong to the same edge). If the number of left and right rays is the same ($r_l = r = r_r$), the equation is simpler:

$$h = r_l + r_r - 2e = 2(r - e).$$

[5]The Euler formula $v - e + f = 2$ for polyhedra, which can be projected to sphere does not apply to pieces of twisty puzzles: the face and edge pieces need not exists: two vertex pieces may be adjacent without an edge between them.

4 Polyhedra suitable for sun puzzles

Figure 8: Polyhedron dual to one of Big Dipper.

The most often used polyhedra for twisty puzzles are regular ones. There are only 5 regular polyhedra, and we can easily confirm this:

Triangular face has by (1) the angle of $180(1 - 2/3) = 60°$. To not exceed $360°$, a vertex can have:

- 3 faces: $60 \times 3 = 180°$ — *Tetrahedron*,
- 4 faces: $60 \times 4 = 240°$ — *Octahedron*,
- 5 faces: $60 \times 5 = 300°$ — *Icosahedron*,
- 6 faces: $60 \times 6 = 360°$ degenerates to planar tiling;

Square face has $180(1 - 2/4) = 90°$, a vertex can have:

- 3 faces: $90 \times 3 = 270°$ — *Cube*,
- 4 faces: $90 \times 4 = 360°$ degenerates to planar tiling;

Pentagonal face has $180(1 - 2/5) = 108°$, a vertex can have:

- 3 faces: $108 \times 3 = 324°$ — *Dodecahedron*;

Hexagonal face has 180(1 - 2/6) = 120°, a vertex can have:

- 3 faces: $120 \times 3 = 360°$ degenerates to planar tiling.

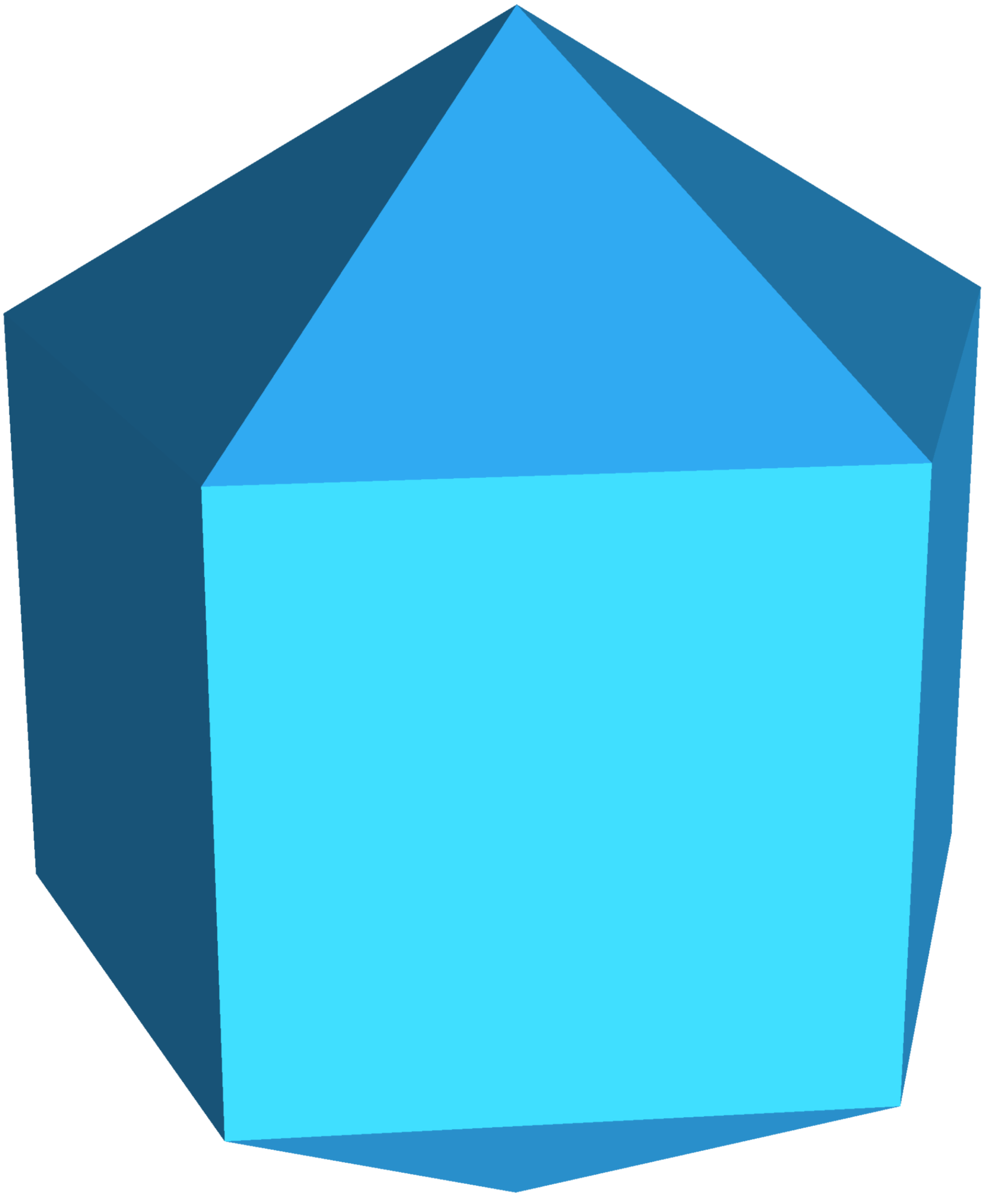

Figure 9: Polyhedron dual to one of Cullinan.

Less often used are Archimedean solids[6]. Their faces are all regular polygons, but not all faces equal to each other. These polyhedra have all equal vertices, but not necessarily regular. The Archimedean polyhedra have a circumscribed sphere and all their faces have circumscribed circles. There are 13 Archimedean polyhedrons and 2 infinite series of prisms and antiprisms. Their mathematical notation is based on vertex configuration and contains as many numbers as many faces meet in vertices, each number names the polygon of the face:

3.6.6 – 60 + 120 + 120 = 300 — *Truncated Tetrahedron*,

3.4.3.4 – 60 + 90 + 60 + 90 = 300 — *Cuboctahedron*,

[6]https://en.wikipedia.org/wiki/Archimedean_solid

3.8.8 – 60 + 135 + 135 = 330 — *Truncated Cube*,

4.6.6 – 90 + 120 + 120 = 330 — *Truncated Octahedron*,

3.4.4.4 – 60 + 90 + 90 + 90 = 330 — *Rhombicuboctahedron*,

4.6.8 – 90 + 120 + 135 = 345 — *Truncated Cuboctahedron*,

3.3.3.3.4 – 60 + 60 + 60 + 60 + 90 = 330 — *Snub Cube*,

3.5.3.5 – 60 + 108 + 60 + 108 = 336 — *Icosidodecahedron*,

3.10.10 – 60 + 144 + 144 = 348 — *Truncated Dodecahedron*,

5.6.6 – 108 + 120 + 120 = 348 — *Truncated Icosahedron*,

3.4.5.4 – 60 + 90 + 108 + 90 = 348 — *Rhombicosidodecahedron*,

4.6.10 – 90 + 120 + 144 = 354 — *Truncated Icosidodecahedron*,

3.3.3.3.5 – 60 + 60 + 60 + 60 + 108 = 348 — *Snub Dodecahedron*,

3.3.3.n – 60 + 60 + 60 + 180 - 360/n = 360(1 - 1/n) — n*-fold Antiprism*,

4.4.n – 90 + 90 + 180 - 360/n = 360(1 - 1/n) — n*-fold Prism*.

Square prism is identical to cube, triangular antiprism is identical to octahedron. Examples of puzzles in form of Archimedean solids are DaYan Gem series. Unfortunately, despite the fact these polyhedra have more elements than regular ones, they do not have a larger combinatorial symmetry. For example, VeryPuzzle Tuttminx has the form of truncated icosahedron with 32 faces, 90 edges and 60 vertices. Still, its combinatorial symmetry is:

Pentagonal faces: 12 × 5 = 60,

Hexagonal faces: 20 × 3 = 60,

Asymmetric edges between pentagonal and hexagonal faces: 60 × 1 = 60,

Symmetric edges between hexagonal faces: 30 × 2 = 60,

Vertices: 60 × 1 = 60.

So, its combinatorial symmetry equals to that of icosahedron 60.

Polyhedra dual to Archimedean, including prisms and antiprisms, are Catalan solids[7]. They have inscribed sphere and each face has inscribed circle:

Triakis Tetrahedron is dual to Truncated Tetrahedron (3.6.6),

Rhombic Dodecahedron is dual to Cuboctahedron (3.4.3.4),

Triakis Octahedron is dual to Truncated Cube (3.8.8),

Tetrakis Hexahedron is dual to Truncated Octahedron (4.6.6),

Deltoidal Icosi-Tetrahedron is dual to Rhombicuboctahedron (3.4.4.4),

Disdyakis Dodecahedron is dual to Truncated Cuboctahedron (4.6.8),

[7] https://en.wikipedia.org/wiki/Catalan_solid

Pentagonal Icosi-Tetrahedron is dual to Snub Cube (3.3.3.3.4),

Rhombic Triacontahedron is dual to Icosidodecahedron (3.5.3.5),

Triakis Icosahedron is dual to Truncated Dodecahedron (3.10.10),

Pentakis Dodecahedron is dual to Truncated Icosahedron (5.6.6),

Deltoidal Hexecontahedron is dual to Rhombicosidodecahedron (3.4.5.4),

Disdyakis Triacontahedron is dual to Truncated Icosidodecahedron (4.6.10),

Pentagonal Hexecontahedron is dual to Snub Dodecahedron (3.3.3.3.5),

n-**fold Trapezohedron** is dual to n-fold Antiprism (3.3.3.n),

n-**fold Bipyramid** is dual to n-fold Prism (4.4.n).

Square bipyramid is identical to octahedron, triangular trapezohedron is identical to cube. All dihedral angles (angles between adjacent face planes) of Catalan solids are equal. Together with regular polyhedra, Catalan solids are suitable for sun twisty puzzles. For example, Sky Eyes has the form of Rhombic Dodecahedron.

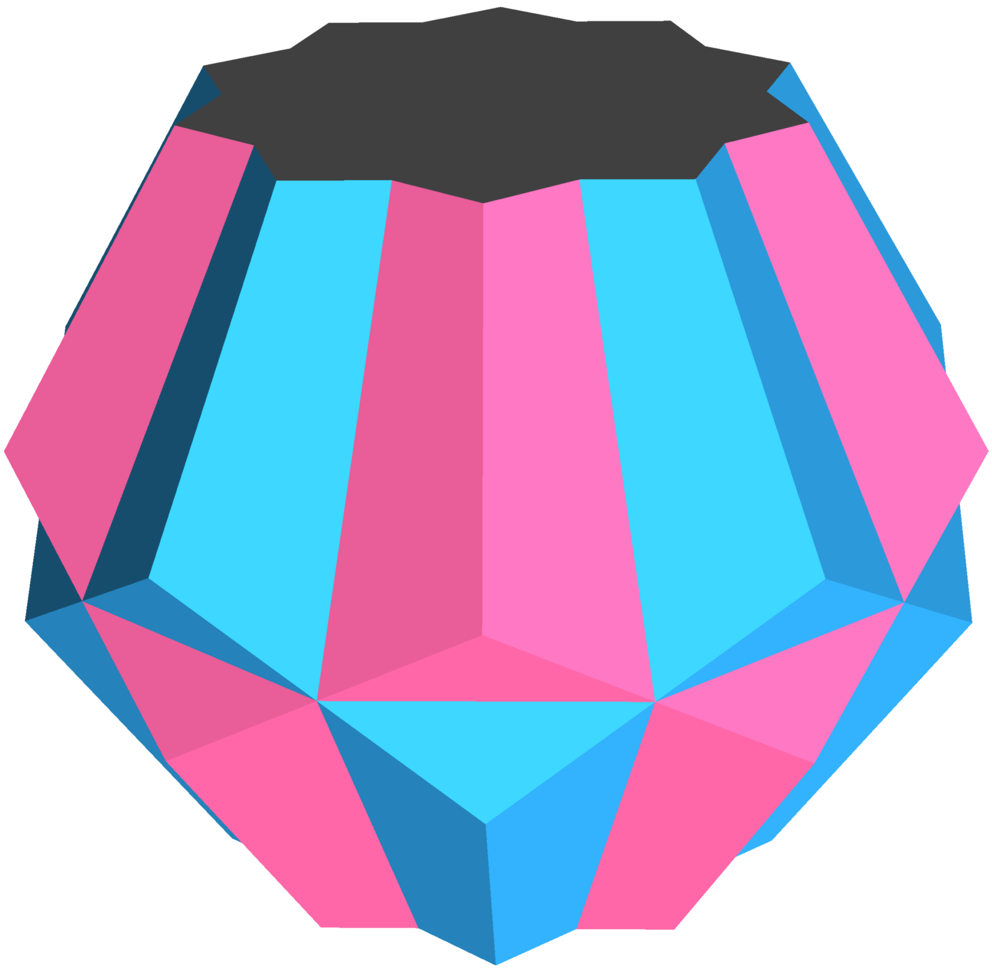

Figure 10: Face intersections of dodecahedra rotated by half face angle.

There are also other solids, whose essential properties are suitable for construction of sun puzzles. They are dual to polyhedra, whose all faces are regular polygons and their vertex defects are equal across all vertices. Unlike Catalan solids, their faces are not always equal among them. However, their vertex configuration are regular, dihedral angles are equal, all faces have inscribed circles and all their radii are equal. For example, Figure 8 shows the polyhedron dual to that of Big Dipper. Its vertex configurations are 3.4.3.4 and 3.3.4.4, obviously all its vertex defects are the same. Figure 9 presents the polyhedron dual to that of Cullinan. Its vertex configurations are 3.3.3.3.3 and 3.3.4.4, and the vertex defects are all equal:

$$60 + 60 + 60 + 60 + 60 = 300 = 60 + 60 + 90 + 90.$$

5 Preparing stickers and discovering face shapes

The cut surfaces of sun puzzles are almost completely defined by geometry of half-twist. Figure 10 presents two dodecahedra rotated by 36° (half of central angle of pentagon, 72°) one to another around the face axis. You can see a well defined sections of sides of pentagon on gray faces as well as cut line of intersected blue and pink faces. The process of finding surface cut lines is presented in Figure 11 using regular pentagon as example with axis order is 10 (1st row left).

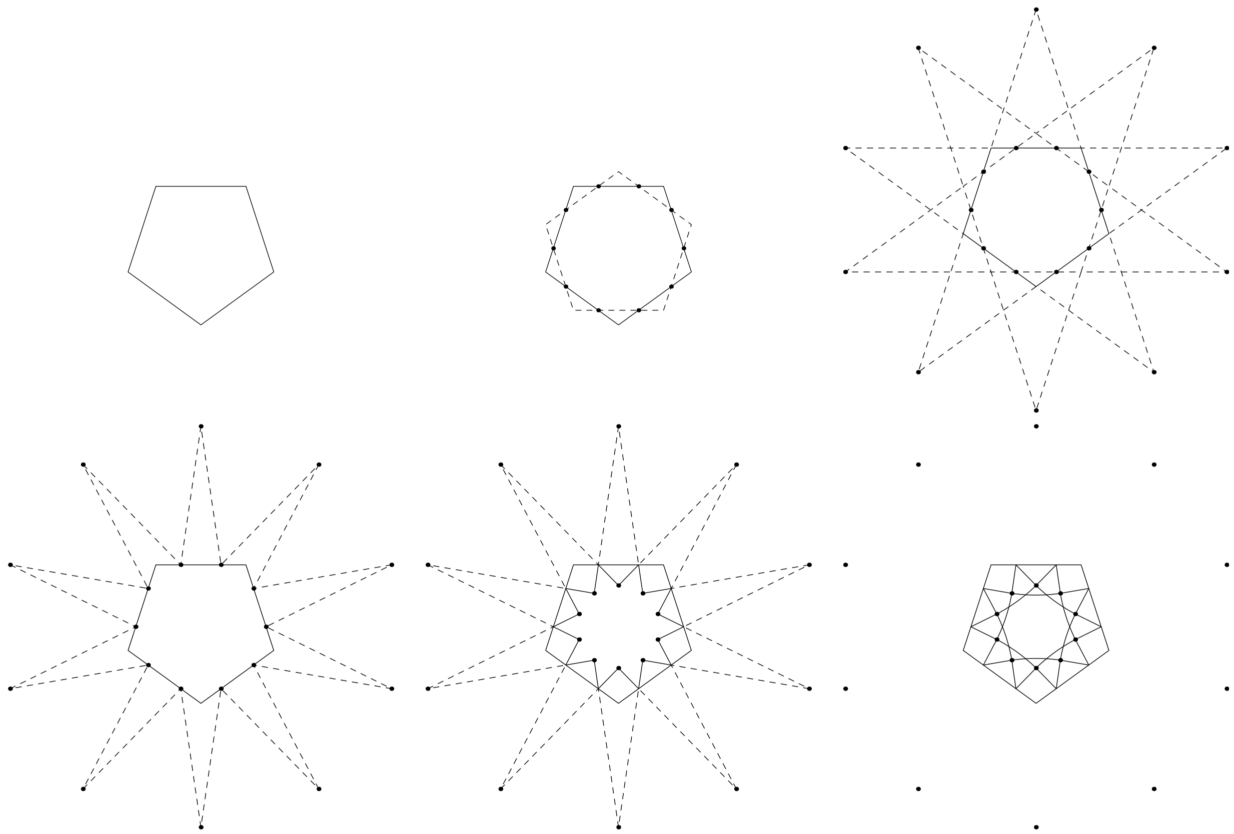

Figure 11: Preparing stickers.

First, prepare a $n/2$-gon, or compound of 2 regular polygons. In our case, there are 2 regular pentagons. Mark the intersection points on polygon sides, 2 points on each side (1st row middle). Then continue the polygon sides until new intersection, they form a $n/4$-gon (1st row right). In our case these are 2 5/2-gons. Mark new intersection points. Connect these new points with nearest intersection points on sides of initial polygon we marked earlier (2nd row left). Continue each line in interior of polygon until new intersection (2nd row middle). Now we have well defined cuts of stickers for vertex and edge pieces. These cuts are straight–lined. The rest of cuts are more free, but it is a good idea to use smooth joint of obtained cuts. A practical way to achieve this is to use circle arcs whose centers are the new points we obtained outside the initial polygon (2nd row right). The stickers for rays, petals and centers are obtained this way.

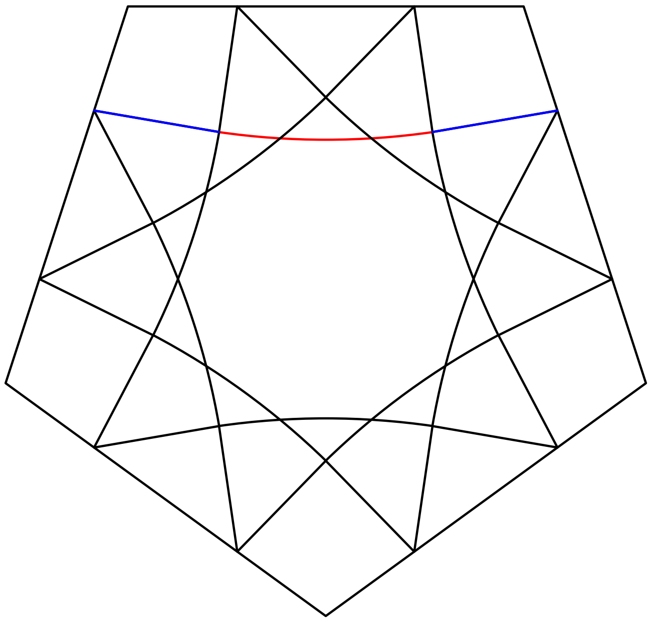

Figure 12: Cut surface of polyhedron.

When transforming the cut line into the surface of revolution, straight lines become surface of hyperboloid of one sheet and circle arc becomes spheric surface smoothly connected with hyperboloid. Figure 12 presents the final result. Hyperboloid cut is marked with blue and spheric with red colors. The ∪-curvature of line is typical for axes order $n > 8$. When $n = 8$ (Sun Cube) all cut lines are straight and all cut surfaces are planar. For $n < 8$ we obtain ∩-curved lines instead.

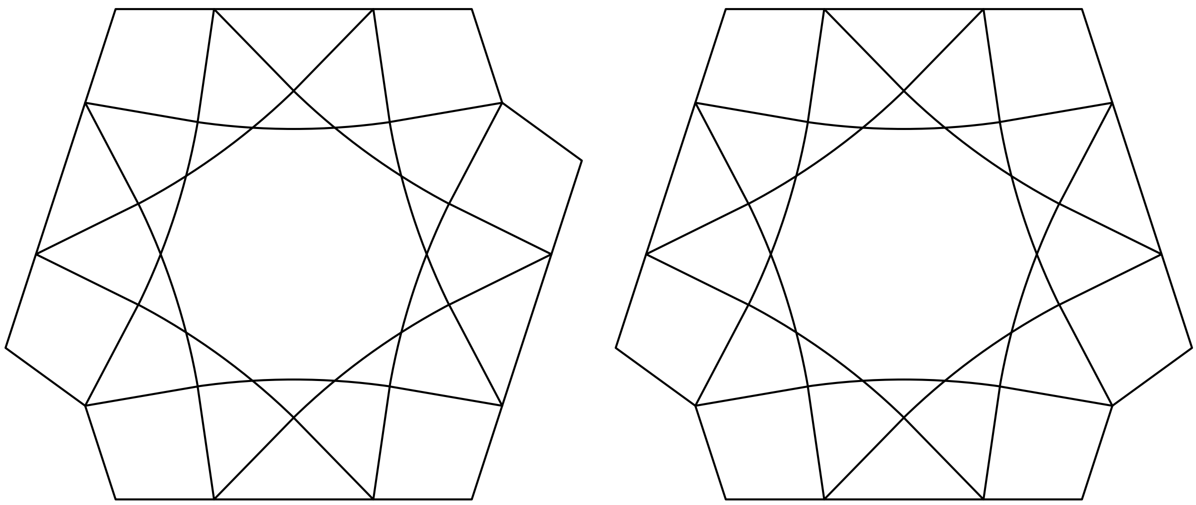

Figure 13: Different arrangements of stickers for 10-fold axis.

We can deduce all possible face shapes just by trying different arrangements of vertex and edge stickers. While doing this, we need to keep in mind several limitations:

1. An edge sticker should be placed between two vertex stickers;

2. Vertex stickers can be alone or two together, but not more.

Figure 12 shows balanced sticker configuration, with edge and vertex stickers placed intermittently. Figure 13 shows sticker configurations with 4 edges, 2 lone vertices and 2 vertex clusters. If we try to use only 3 edge stickers, we end up with 3 consecutive vertex stickers. These vertex configurations are present in Sky Eyes, Big Dipper and Cullinan. The left arrangement indicates the rhombic shape of face while the right one indicates the trapezoidal shape.

6 Finding new puzzles with the algorithm

We now apply the algorithm to describe several sun puzzles. We leave without attention the first 2 rows of Table 1. The axis order 0 has uncertain geometric meaning. The polyhedron with 3 faces is possible to construct in pillow form, and the axis of order 4 has a good symmetry. Still, we concentrate on flat polyhedra.

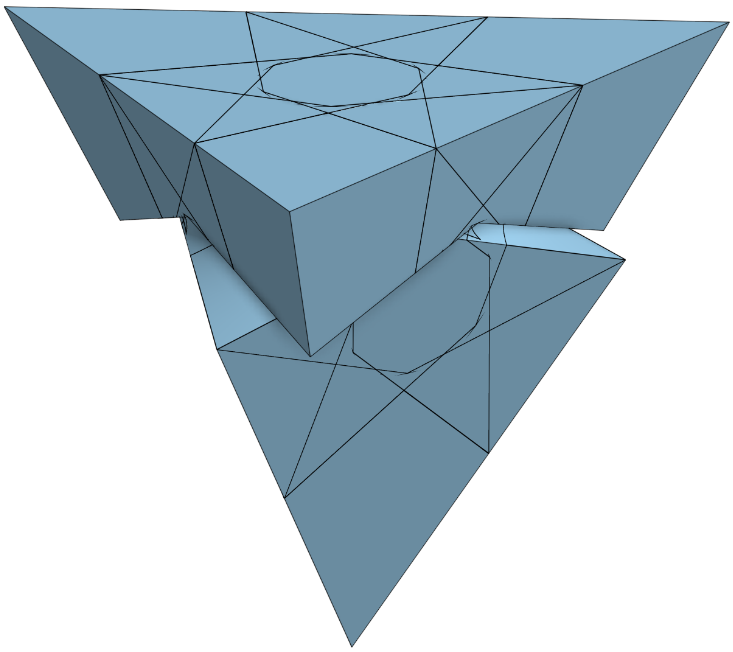

Figure 14: Sun Tetrahedron preliminary design.

We start with polyhedron with 4 faces and axes of order 6. There is only one such polyhedron with flat faces — tetrahedron. Figure 14 presents a preliminary design of it. The face pattern delimited by center, petals and rays is $n/3$-gon. In case of tetrahedron it is 6/3-gon, which splits into 2-gons. Because the split lines are anyway $\cap$-curved, it is possible to design such pillow 2-gons. During designing I was inclined to think the dihedral angle of tetrahedron is too small to serve a sun mechanism. Now, I believe it is possible to find the cut surfaces for the mechanism and to materialize this puzzle.

For 8-fold axes there is Sun Cube. But it is not the only puzzle with 6 faces. When we are out of ideas, we always can take some bipyramid or trapezohedron from two infinite series. The triangular bipyramid has 6 faces and can be the basis of sun puzzle as it is shown in Figure 15. All its faces are equal, the puzzle has 2 standalone vertices and 3 clusters of 4 vertices. Its flat and dihedral angles are larger than 90° of Sun Cube, still the design is possible and hopefully the pieces can be designed to be interchangeable with Sun Cube.

Next, we search for 9-fold axes on polyhedron with 8 faces. The first idea is to try octahedron. It is indeed possible as it is shown in Figure 16. The only issue is octahedron has no 3-fold vertices.

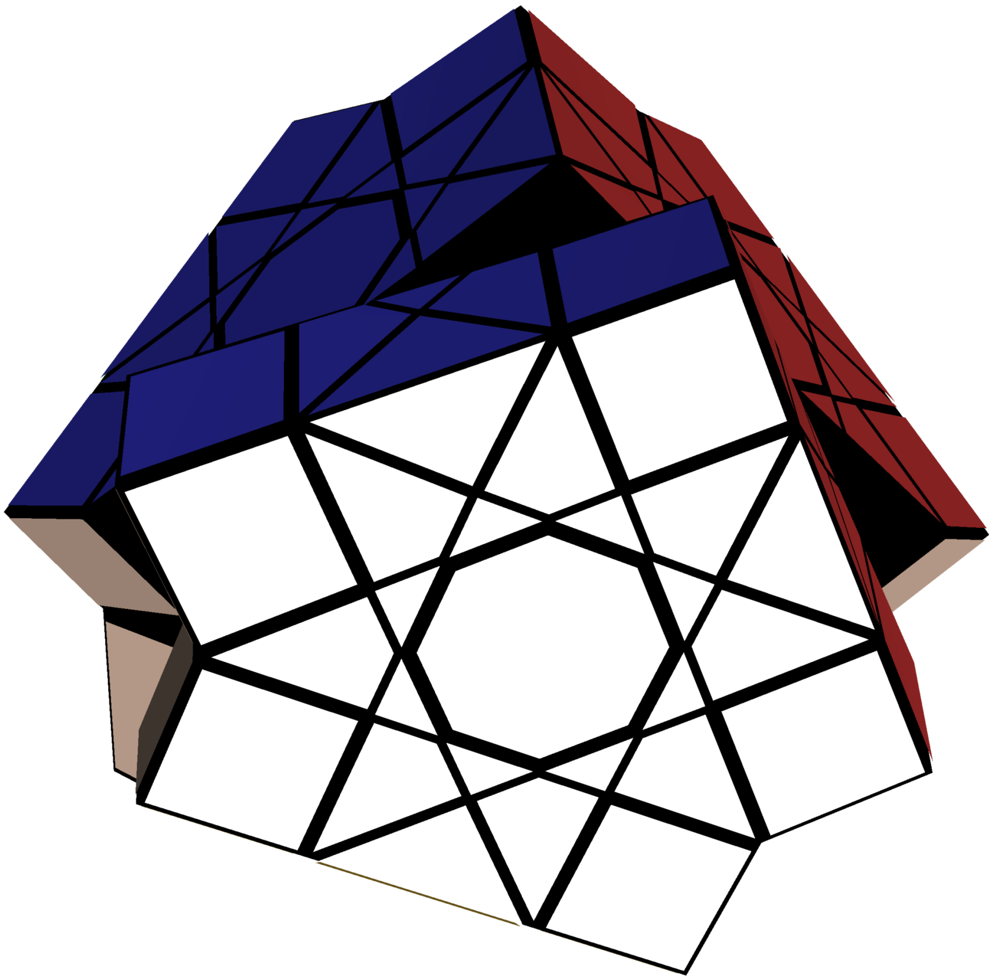

Figure 15: Sun Triangular Bipyramid concept.

So, all its vertices become clusters of 4 vertices of sun puzzle. Due to this fact, there are certain difficulties in computing the exact geometry of this puzzle. Still, hopefully there is no showstopper for designing sun octahedron.

Again, nothing stops us from searching another polyhedra with 8 faces to design sun puzzle with 9-fold axes. And the trapezohedron series contains square trapezohedron. Figure 17 shows its concept. The good news about this shape is its real flat and dihedral angles are very close to ideal ones, necessary for 9-fold combinatorial symmetry, which can be observed in Table 3. This puzzle also has all equal faces. It has 8 single vertices and 2 clusters of 4 vertices.

Going further for puzzles with 10-fold axes is probably more interesting and may result in puzzles of many different shapes. Because mf8 already presented us 4 such puzzles, we leave this task for enthusiasts. The more axes a polyhedron have, the more diversity we can expect in polyhedra suitable for sun puzzles. For example, among Catalan solids, the following have 12 faces: triakis tetrahedron and rhombic dodecahedron (Sky Eyes is based on it).

Finally, we reach the last possible 11-fold axes. The equation 3 says these puzzles should have 24 faces. It turns out that such polyhedra do exist among Catalan solids. One of them is pentagonal icosi–tetrahedron. Figure 18 presents sun puzzle based on this polyhedron. All its faces are equal. It has 32 standalone vertices and 6 clusters of 4 vertices. The polyhedron is chiral, that it, its mirror polyhedron is different, so here we described 2 almost identical puzzles. Table 3 says its flat and dihedral angles are very close to theoretically necessary for 11-fold symmetry, which means this puzzle should be easy to turn, but not so easy to solve.

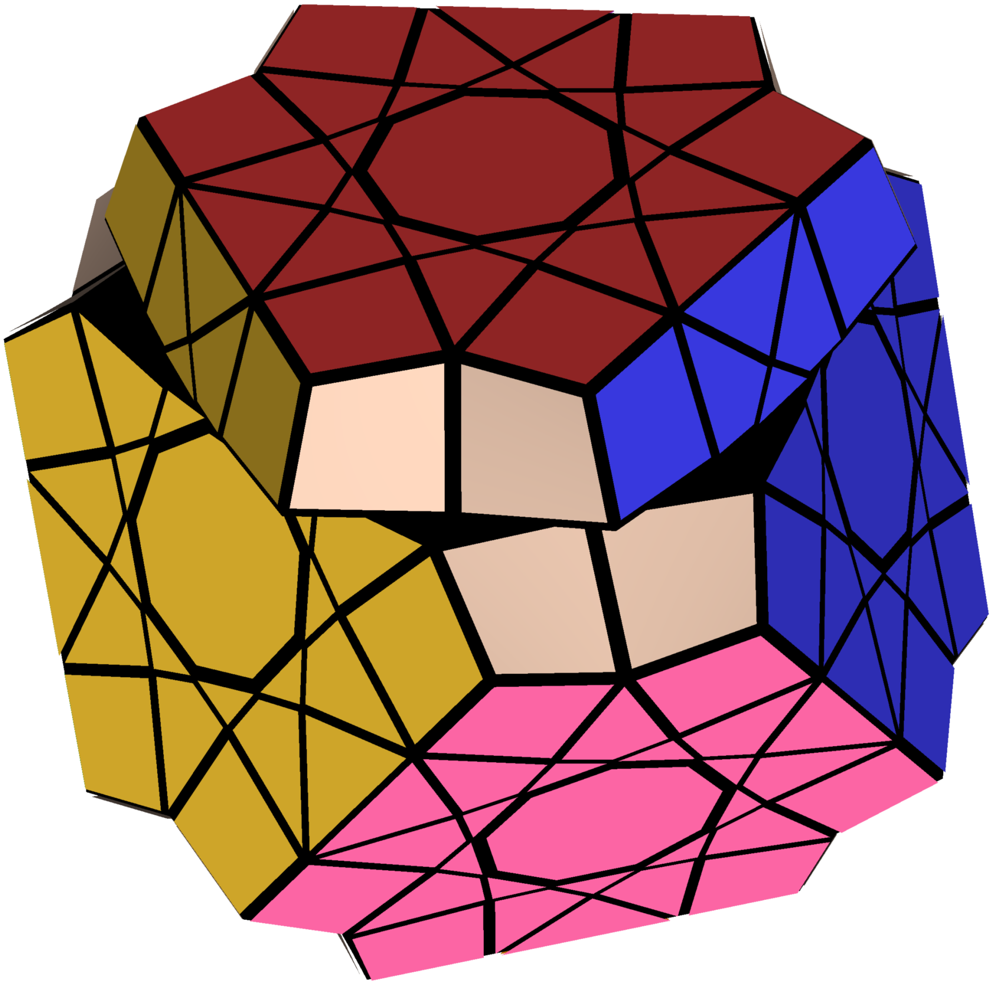

Figure 16: Sun Octahedron concept.

Again, there are also other suitable polyhedra with 24 faces. Another Catalan solid is deltoidal icosi–tetrahedron. A preliminary design of sun puzzle based on it is shown in Figure 19. Its flat and dihedral angles are slightly different from those of pentagonal icosi–tetrahedron, still they are very close to ideal ones. As pentagonal icosi–tetrahedron, deltoidal icosi–tetrahedron has all faces equal. But unlike pentagonal polyhedron, deltoidal puzzle has 8 standalone vertices and 18 clusters of 4 vertices.

Of course, there are also other polyhedra with 24 faces. From Catalan list we can mention triakis octahedron, tetrakis hexahedron, deltoidal icositetrahedron (already considered) and pentagonal icositetrahedron (also considered earlier). But there is one polyhedra, which is not Catalan one, which can be the base for sun puzzle, it is quasi–deltoidal icosi–tetrahedron — Figure 20. It can be obtained from deltoidal icosi–tetrahedron by cutting it into halves, twisting one by 45° and gluing the halves again. It can be imagined easier if we consider its dual polyhedron — pseudo–rhombicuboctahedron (Figure 21), which can be obtained from rhombicuboctahedron in similar way as its dual. This puzzle is very similar to previous one.

This short list of concepts presents the most obvious sun puzzle, which immediately arise when applying the algorithm of their search. Still, we already discovered 7 essentially different sun twisty puzzles. Table 2 shows details about different pieces of these as well as of existent sun puzzles.

The table divides all the puzzles in categories by axes order. Besides the order we can observe two more details common for puzzles inside a category: the number of faces f and the number of vertices together with twice the number of edges $v + 2e$. Any of these can describe the category of

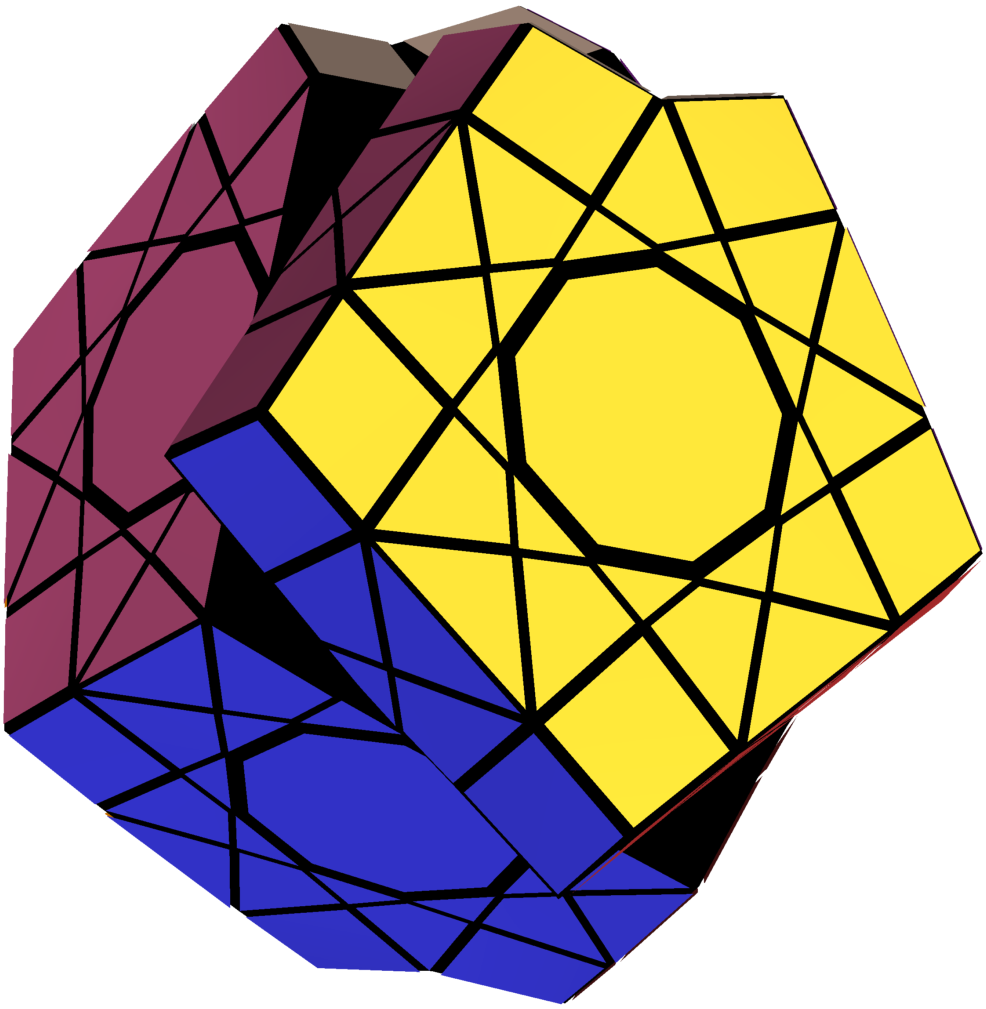

Figure 17: Sun Square Trapezohedron concept.

puzzles. We can observe that $v + 2e$ (the puzzle "price") from penultimate column is always multiple to 16. So, we can introduce the puzzle complexity (the last column) as this number divided by 16.

The list of sun puzzles is not exhaustive. Its purpose if to show that the axes configurations found by algorithm *do exist*, and that usually there are *several* sun puzzles with given axes order and number of faces.

As we can see, the algorithm only searches restricted subset of polyhedra. The first artificial limitation is deliberate exclusion of 5-verties from consideration. This way we constructed a nice theory, but 5-vertex configuration also needs to be analyzed. Then, the theory does not search for polyhedra with axes order 5 and 7, because no polyhedra with equal dihedral angles and these axes exist. But such sun puzzles exist with different dihedral angles! One example is Lapis[8] which has 6 axes of order 7 — Figure 22. On the other hand, it is unclear the possibility to construct (possibly pillow) puzzle with 3 axes of order 4. Is it also unclear if the axes of order 0 has any geometric meaning as well as the form of polyhedron with only 2 faces.

Including also 5-fold axes can potentially lead to interesting consequences. For example, tetrahedron has 3-fold vertices and its sun puzzle has axes of order 6. Octahedron has 4-fold vertices and its sun puzzle has axes of order 9. Icosahedron has 5-fold axes and its sun puzzle, if exists, would have axes of order 12. Currently, the theory declares such puzzles 2D. But, if this is possible, the

[8] By Jeong Min Kim, https://twistypuzzles.com/app/museum/museum_showitem.php?pkey=10250, https://twistypuzzles.com/app/museum/museum_showitem.php?pkey=10251.

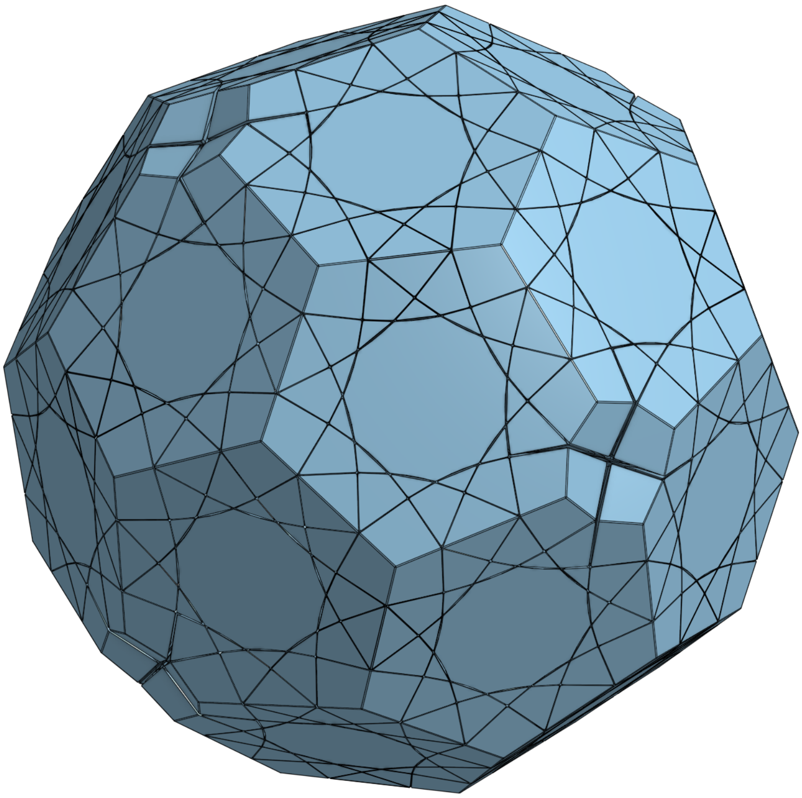

Figure 18: Sun Pentagonal Icosi–Tetrahedron design.

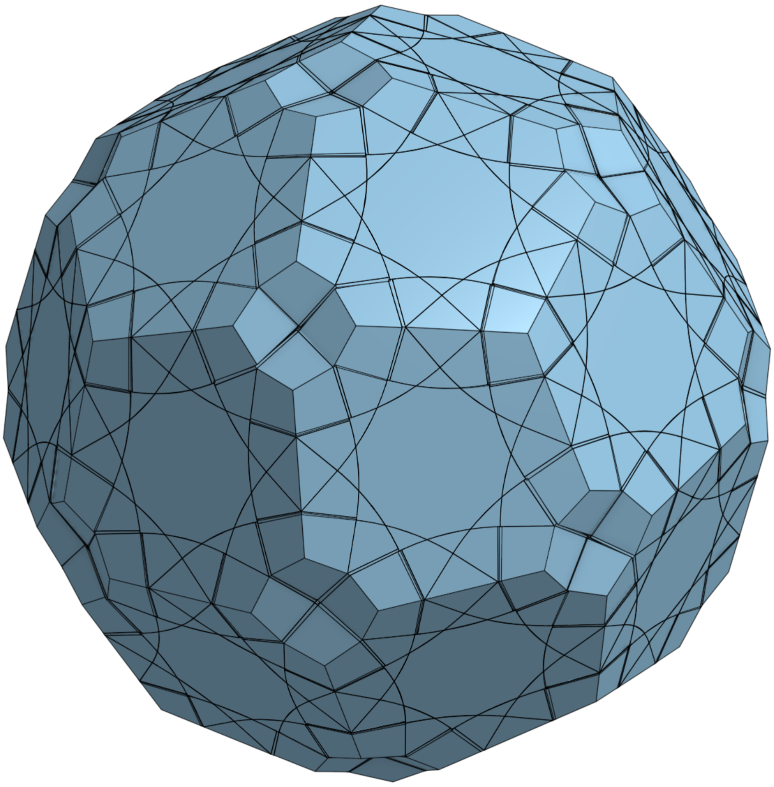

Figure 19: Sun Deltoidal Icosi–Tetrahedron preliminary design.

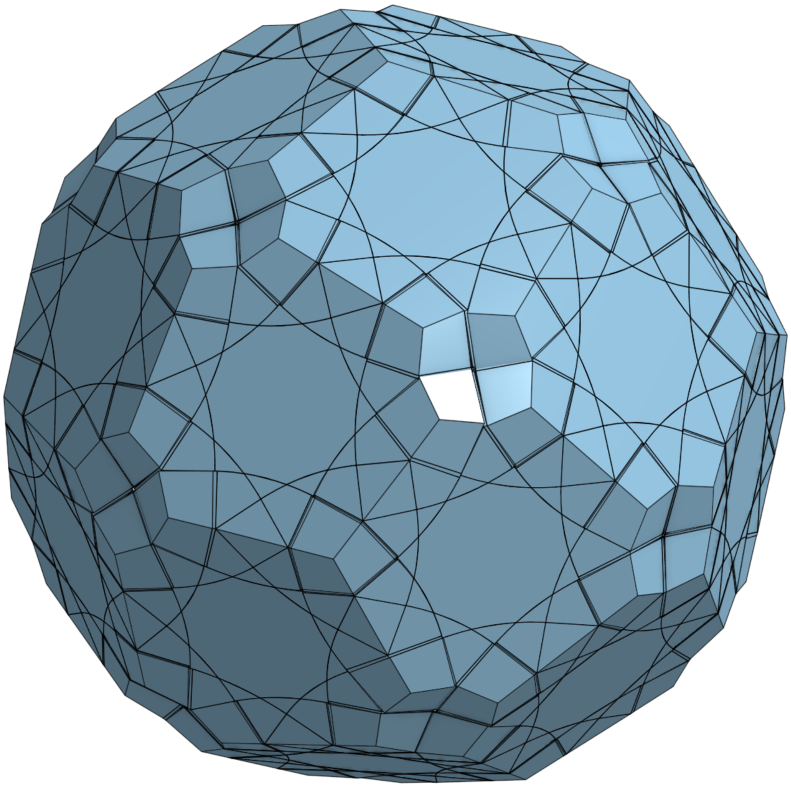

Figure 20: Sun Quasi–Deltoidal Icosi–Tetrahedron preliminary design.

Table 2: Elements of sun twisty puzzles.

Sun Puzzle	Number of pieces						$v + 2e$	Complexity
	f	v	e	p	r	h		
Tetrahedron	4	4	6	24	12	12	16	1
Sun Cube	6	8	12	48	24	24	32	2
Triangualar Bipyramid	6	14	9	48	24	30	32	2
Octahedron	8	24	12	72	36	48	48	3
Square Trapezohedron	8	16	16	72	36	40	48	3
Sunminx	12	20	30	120	60	60	80	5
Cullinan	12	30	25	120	60	70	80	5
Sky Eyes	12	32	24	120	60	72	80	5
Big Dipper	12	32	24	120	60	72	80	5
Pentagonal Icosi–Tetrahedron	24	56	60	264	132	144	176	11
Deltoidal Icosi–Tetrahedron	24	80	48	264	132	168	176	11
Quasi–Deltoidal Icosi–Tetrahedron	24	80	48	264	132	168	176	11

theory would be essentially extended.

7 Other questions

While we described the math behind sun twisty puzzles, we only used calculations on integer numbers. Nowhere we checked the face angles or dihedral angles of polyhedra. This is done intentionally

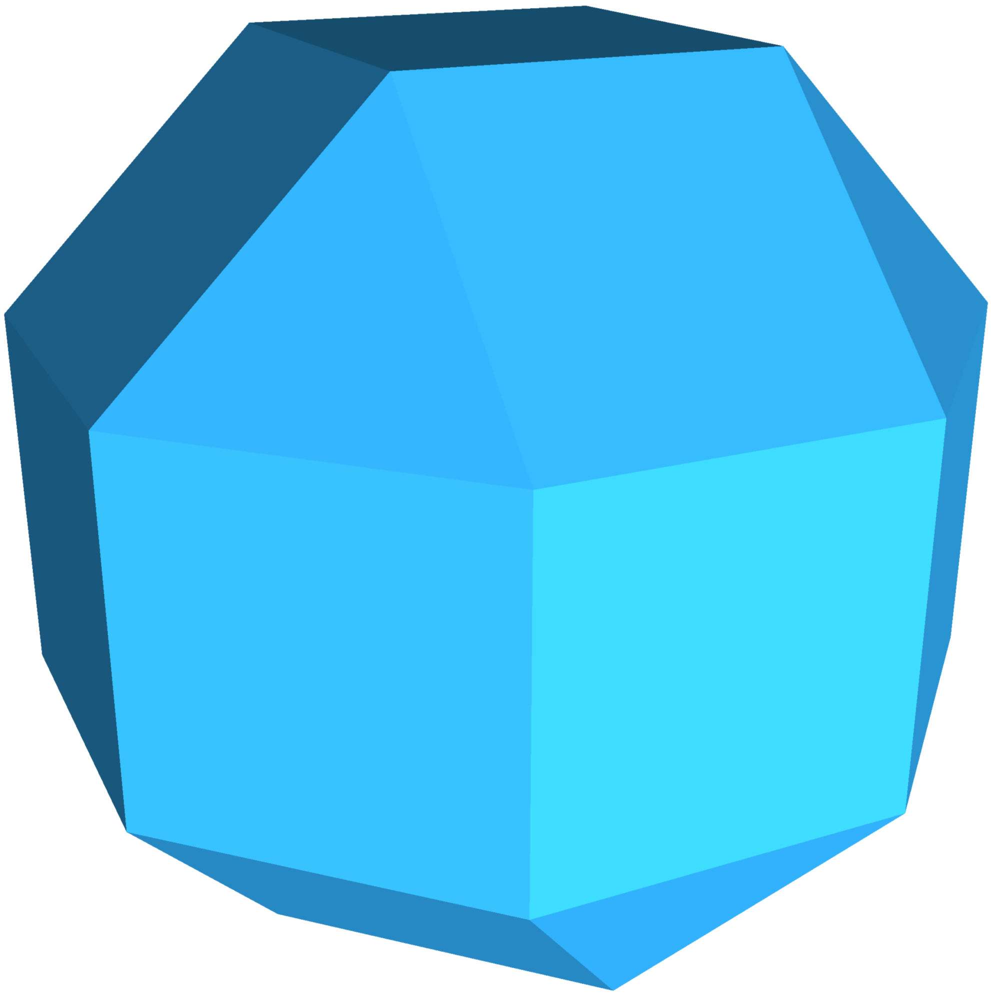

Figure 21: Pseudo–Rhombicuboctahedron dual to Quasi–Deltoidal Icosi–Tetrahedron.

to show that the properties of polyhedra to be suitable for sun twisty puzzles is topological, or combinatorial, but not geometrical. Still, it is helpful to take a look at these angles on sun twisty puzzles. Table 3 presents flat and dihedral angles of described puzzles as their real values and the values, which are ideal for their axis order. The dihedral angle can be computed from face angle using spherical version of Cosine I Law, the reverse is possible with Cosine II Law. However, having regular 3-vertex configuration, they can be computed by simpler equations:

$$\cos\theta = \frac{\cos\alpha}{1+\cos\alpha}, \quad \cos\alpha = \frac{\cos\theta}{1-\cos\theta}.$$

In case of 4-vertex, the face angle α can be computed from the angles between opposite vertex edges φ, ψ by spherical version of Pythagorean theorem:

$$\cos\alpha = \cos\frac{\varphi}{2}\cos\frac{\psi}{2}.$$

We can observe that the difference between ideal and real values are most of the time not too large. Also, we can observe that the real angles of different polyhedra within the same category are close to each other. The angle values is the result of the theory, not a requirement.

Speaking on inexact geometry, it is perfectly acceptable for mechanical puzzles. When designing a puzzle, the geometry should always be based on real polyhedron geometry, not on ideal values.

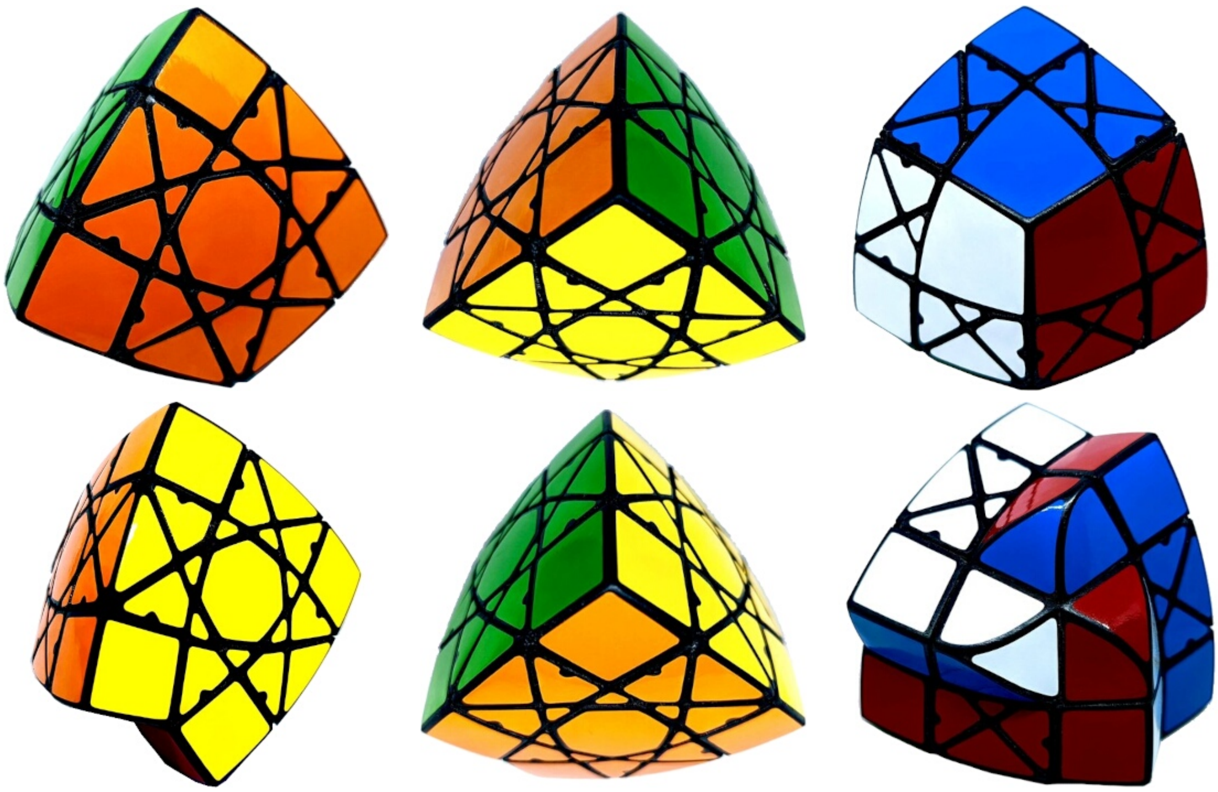

Figure 22: Lapis puzzle with axes of order 7: 1st row – V1, 2nd row – V2.

Table 3: Angles of sun twisty puzzles.

Sun Puzzle	Flat angle α		Dihedral angle θ		Tan $\theta/2$
	Ideal	Real	Ideal	Real	
Tetrahedron	60°	60°	70.53°	70.53°	$1/\sqrt{2}$ = 0.71
Sun Cube	90°	90°	90°	90°	1
Triangualar Bipyramid	90°	97.18°	90°	98.21°	1.15
Octahedron[9]	100°	104.48°	102.13°	109.47°	$\sqrt{2}$ = 1.41
Square Trapezohedron	100°	101.95°	102.13°	105.14°	1.31
Sunminx	108°	108°	116.57°	116.57°	$(1+\sqrt{5})/2$ = 1.62
Cullinan[10]	108°	108°	116.57°	116.57°	$(1+\sqrt{5})/2$ = 1.62
Sky Eyes	108°	109.47°	116.57°	120°	$\sqrt{3}$ = 1.73
Big Dipper	108°	109.47°	116.57°	120°	$\sqrt{3}$ = 1.73
Pentagonal Icosi–Tetrahedron	114.55°	114.81°	135.28°	136.31°	2.49
Deltoidal Icosi–Tetrahedron	114.55°	115.26°	135.28°	138.12°	2.61
Quasi–Deltoidal Icosi–Tetrahedron	114.55°	115.26°	135.28°	138.12°	2.61

The only exception is central face piece, which needs to be an ideal n-gon and should touch petal pieces having near-miss geometry but sufficiently tight.

Sun puzzles of the same complexity can be designed to have interchangeable pieces. For this, we should ensure not only their shape are similar but also their size is comparable. Catalan solids have inscribed sphere and each face has inscribed circle. The radius of inscribed in face circle r of

[9]Octahedron has no 3-vertices. The flat angles are computed from dihedral angles if 3-vertices would exist.
[10]Cullinan has not all dihedral angles equal. The values are for standalone 3-vertices.

puzzles with interchangeable pieces should match. Due to slightly different dihedral angles, in this case the radii of inscribed spheres R differ. More exactly:

$$\frac{R}{r} = \tan\frac{\theta}{2},$$
$$r = \frac{R}{\tan\theta/2}.$$

So, if two puzzles have interchangeable pieces, they have $r_1 = r_2 = r$, and:

$$\frac{R_1}{\tan\theta_1/2} = r_1 = r_2 = \frac{R_2}{\tan\theta_2/2},$$
$$\frac{R_1}{R_2} = \frac{\tan\theta_1/2}{\tan\theta_2/2}.$$

So, the sizes of puzzles of the same complexity compare as $\tan\theta/2$ of their polyhedra. This value is shown for each puzzle in the last column of Table 3.